

# PARTIAL MONOID ACTIONS AND A CLASS OF RESTRICTION SEMIGROUPS

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**ABSTRACT.** We study classes of proper restriction semigroups determined by properties of partial actions defining them. Of particular interest is the class arising from homomorphisms from a monoid  $T$  to the Munn monoid of a semilattice  $Y$ . We call elements of this class ultra  $F$ -restriction monoids and show that it coincides with the class of monoids  $Y *_m T$  considered by Fountain, Gomes and Gould. We then construct an embedding of ultra  $F$ -restriction monoids for which the base monoid  $T$  is free into  $W$ -products of semilattices by monoids. This generalizes Szendrei's construction of embedding of free restriction monoids into  $W$ -products and can be effectively applied in a different setting. We thus strengthen two recent embedding-covering results involving  $W$ -products by Szendrei and provide new and simpler proofs of these results.

## 1. INTRODUCTION

Restriction semigroups, also known as weakly  $E$ -ample semigroups, are non-regular generalizations of inverse semigroups. These are semigroups with two additional unary operations which mimic the operations  $a \mapsto a^{-1}a$  and  $a \mapsto aa^{-1}$  on an inverse semigroup. Various aspects of restriction semigroups and their one-sided analogues have been extensively studied in the literature, see, e.g., [3, 5, 9] and references therein.

Proper restriction semigroups are analogues of  $E$ -unitary inverse semigroups which play a central role in the theory of inverse semigroups and its applications. Generalizing corresponding results for inverse and ample semigroups [18, 11, 12], Cornock and Gould [1] gave a structure theorem for proper restriction semigroups in terms of double partial actions of monoids on semilattices. This can be readily reformulated in terms of only one partial action, since each of the two is determined by the other one. In the present paper we consider classes of proper restriction semigroups determined by the properties of this partial action. We show that the partial action is strong or antistrong if and only if the restriction semigroup satisfies a technical condition arising in [1].  $W$ -products of semilattices by monoids correspond to the situation when the partial action is an action. More importantly, we single out a class of partial actions which correspond to dual prehomomorphisms which are in fact homomorphisms. At first sight, this requirement looks

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rather restrictive, since for partial actions of groups it reduces a partial action to an action. However, quite surprisingly, free restriction monoids and semigroups belong to this class (but not free inverse monoids and semigroups). We call elements of this class ultra proper restriction semigroups. In fact, its subclass for which the defining homomorphism has its range in the Munn monoid of the semilattice is of even more interest. We call the arising proper restriction monoids ultra  $F$ -restriction and show that they are precisely the monoids  $Y *_m T$  introduced by Fountain, Gomes and Gould in [3].

Based on the ideas from [17] and [16] we next construct a globalization of a strong partial action defining a proper restriction semigroup and obtain a McAlister-type theorem. We then specialize this construction to partial actions defining ultra  $F$ -restriction monoids  $M(T, Y)$  where the base monoid  $T$  is free. As a result, we obtain a semilattice  $X$  and an action of  $T$  on this semilattice such that the  $W$ -product  $W(T, X)$  can be formed and moreover, the initial monoid  $M(T, Y)$  embeds into  $W(T, X)$ . This generalizes Szendrei's embedding of the free restriction monoid into a  $W$ -product [19] and, importantly, can be effectively applied in a different setting which we now outline. Consider an  $A$ -generated restriction monoid  $S$  and let  $T = A^*$  be the  $A$ -generated free monoid. There is a partially defined action of  $T$  on the semilattice of projections  $Y$  of  $S$  which defines an ultra  $F$ -restriction (and even ample) monoid  $M(T, Y)$  which covers  $S$ . If  $S$  is a restriction semigroup, a modification of this construction produces an ultra proper restriction semigroup  $M(T, Y)$  which covers  $S$  and is a restriction subsemigroup of the ultra  $F$ -restriction monoid  $M(T, Y^1)$ . The globalization construction enables us to embed  $M(T, Y)$  into a  $W$ -product  $W(T, X)$  and thus strengthen two embedding-covering results by Szendrei [19, 20] and provide new and simpler proofs of these results. The first result [19] states that any restriction semigroup  $S$  has a proper ample cover embeddable into a  $W$ -product, and our argument proving this is that  $M(T, Y)$  provides such a cover. The second result [20] states that any restriction semigroup can be embedded into an almost left factorizable restriction semigroup, that is, a quotient of a  $W$ -product. To show this we construct a (projection separating) congruence  $\kappa$  on  $W(T, X)$  which extends the congruence on  $M(T, Y)$  mapping it onto  $S$ , so that  $S$  embeds into  $W/\kappa$ .

## 2. PRELIMINARIES

**2.1. Restriction semigroups.** Here we recall the definition and basic properties of restriction semigroups. Further details, including a different approach to restriction semigroups, via generalized Green's relations, can be found in [5, 9].

Let  $S$  be a semigroup and  $E$  be a commutative subsemigroup of idempotents of  $S$ . The semigroup  $S$  is called a *restriction semigroup with respect to  $E$*  if there are two functions  $\lambda, \rho : S \rightarrow S$  such that the following axioms are satisfied:

- (RS1) for all  $a \in S$ :  $\lambda(a) \in E$  and  $\rho(a) \in E$ ;
- (RS2) for all  $a \in E$ :  $\lambda(a) = \rho(a) = a$ ;
- (RS3) for all  $a \in S$ :  $\rho(a)a = a$  and  $a\lambda(a) = a$ ;
- (RS4) for all  $a, b \in S$ :  $\lambda(ab) = \lambda(\lambda(a)b)$  and  $\rho(ab) = \rho(a\rho(b))$ ;
- (RS5) for all  $a \in S$  and  $e \in E$ :  $ea = a\lambda(ea)$  and  $ae = \rho(ae)a$ .

The above axioms are used throughout the text, possibly without reference. The elements of  $E$  are called *projections* of  $S$ . If the semilattice  $E$  is understood, we refer to  $S$  just as *restriction semigroup*. We use the notation  $P(S)$  for the semilattice of projections of  $S$ , that is we set  $P(S) = E$ . Note that a projection is necessarily an idempotent, but a restriction semigroup may contain idempotents which are not projections.

An important aspect of analogy between restriction and inverse semigroups is that restriction semigroups are equivalent to inductive categories in a similar way as are inverse semigroups equivalent to inductive groupoids [13, 10].

A restriction semigroup  $S$  is called *ample* if for all  $a, b, c \in S$ :

$$ac = bc \Rightarrow a\rho(c) = b\rho(c); \quad ca = cb \Rightarrow \lambda(c)a = \lambda(c)b.$$

Under the correspondence between restriction semigroups and inductive categories, ample semigroups correspond to *cancellative* inductive categories.

Restriction semigroups are conveniently looked at as algebras of the signature  $(2, 1, 1)$  where the binary operation is the multiplication and the unary operations are given by the functions  $\lambda$  and  $\rho$ . Homomorphisms, congruences, subalgebras etc. are understood in this signature. Thus, by definition, a homomorphism of restriction semigroups is required to preserve the multiplication and the functions  $\lambda$  and  $\rho$ . For emphasis, we will sometimes use the terms  $(2, 1, 1)$ -morphism,  $(2, 1, 1)$ -congruence,  $(2, 1, 1)$ -subalgebra, etc. If we discuss restriction monoids we consider them as  $(2, 1, 1, 0)$ -algebras and similar remarks apply. Notice that if  $S$  is a restriction monoid with identity element 1, then from (RS4) we obtain that  $\lambda(1) = \rho(1)$  is the maximum projection which yields  $\lambda(1) = \rho(1) = 1$ .

Let  $S$  be a restriction semigroup and  $E = P(S)$ . For  $a, b \in S$  we set  $a \leq b$  provided that there is  $e \in E$  with  $a = eb$ . This relation is a partial order called the *natural partial order* on  $S$ . The following properties of restriction semigroups related to the partial order will be used throughout the paper, possibly without reference.

- (RS6)  $a \leq b$  if and only if there is some  $f \in E$  such that  $a = bf$ .
- (RS7)  $a \leq b$  if and only if  $a = b\lambda(a) = \rho(a)b$ .
- (RS8) Let  $e \in E$  be such that  $ea = a$  ( $ae = a$ ). Then  $e \geq \rho(a)$  (respectively,  $e \geq \lambda(a)$ ).
- (RS9)  $a \geq ae, ea$  for any  $e \in E$ .
- (RS10) The order  $\geq$  is compatible with the multiplication, that is, if  $a \geq b$  then  $ac \geq bc$  and  $ca \geq cb$ .

Let  $\sigma$  denote the least congruence on a restriction semigroup  $S$  which identifies all elements of  $P(S)$ . It is well known that the following statements are equivalent: (i)  $a \sigma b$ ; (ii) there is  $e \in E$  such that  $ea = eb$ ; (iii) there is  $e \in E$  such that  $ae = be$ .

A restriction semigroup  $S$  is called *proper* if the following two conditions hold:

$$\begin{aligned} &\text{for any } a, b \in S : \text{ if } \lambda(a) = \lambda(b) \text{ and } a \sigma b \text{ then } a = b; \\ &\text{for any } a, b \in S : \text{ if } \rho(a) = \rho(b) \text{ and } a \sigma b \text{ then } a = b. \end{aligned}$$

A  $(2, 1, 1)$ -morphism  $\varphi : T \rightarrow S$  of restriction semigroups is called *projection separating* if  $\varphi(e) \neq \varphi(f)$  for any two distinct projections  $e, f$  of  $T$ . A restriction semigroup  $T$  is

called a *cover* of a restriction semigroup  $S$  if there is a surjective projection separating  $(2, 1, 1)$ -morphism  $\varphi : T \rightarrow S$ .

**2.2. Proper restriction semigroups and the compatibility relation.** We say that elements  $a, b$  of a restriction semigroup  $S$  are *compatible* and denote this by  $a \sim b$  if  $a\lambda(b) = b\lambda(a)$  and  $\rho(b)a = \rho(a)b$ . Compatible order ideals were considered in the literature [4, 20] under the name *permissible sets*. The notion of compatibility extends the corresponding notion for inverse semigroups [14]. The following basic properties of the compatibility relation are easy to verify:

**Lemma 1.**

- (1) If there is  $c \in S$  such that  $c \geq a, b$  then  $a \sim b$ .
- (2) If  $a \sim b$  and  $\lambda(a) = \lambda(b)$  (or  $\rho(a) = \rho(b)$ ) then  $a = b$ .
- (3) The compatibility relation is stable with respect to the multiplication, that is,  $a \sim b$  and  $c \sim d$  imply that  $ac \sim bd$ .

The following result shows that proper restriction semigroups admit a similar characterization via the compatibility relation as  $E$ -unitary inverse semigroups.

**Lemma 2.** For a restriction semigroup  $S$  the following conditions are equivalent:

- (1)  $S$  is proper;
- (2)  $\sim = \sigma$ .
- (3)  $\sim$  is a congruence;
- (4)  $\sim$  is an equivalence relation;

*Proof.* (1)  $\Rightarrow$  (2) Assume that  $S$  is proper. Since, clearly,  $\sim \subseteq \sigma$ , we verify only the reverse implication. Let  $a \sigma b$  and define  $a' = a\lambda(b)$  and  $b' = b\lambda(a)$ . Then  $a' \sigma b'$  and  $\lambda(a') = \lambda(b') = \lambda(a)\lambda(b)$ . Since  $S$  is proper, we obtain  $a' = b'$ . If we define  $a'' = \rho(b)a$  and  $b'' = \rho(a)b$  we similarly get  $a'' = b''$ . It follows that  $a \sim b$ .

The implications (2)  $\Rightarrow$  (3) and (3)  $\Rightarrow$  (4) are obvious.

(4)  $\Rightarrow$  (1) Assume that  $\sim$  is an equivalence relation and let  $a, b \in S$  be such that  $\lambda(a) = \lambda(b)$  and  $a \sigma b$ . Let  $e \in E$  be such that  $ae = be$ . From  $a \sim ae = be \sim b$  it follows that  $a \sim b$ , hence  $a = b$  by Lemma 1(2). It can be similarly shown that  $\rho(a) = \rho(b)$  and  $a \sigma b$  imply  $a = b$ . Thus  $S$  is proper. □

**2.3. Actions.** Let  $T$  be a monoid with identity 1 and  $X$  be a set. A *left action* of  $T$  on  $X$  is a map  $T \times X \rightarrow X$ ,  $(t, x) \mapsto t * x$ , such that  $1 * x = x$  for all  $x \in X$  and  $s * (t * x) = (st) * x$  for all  $s, t \in T$  and  $x \in X$ . A *right action* of  $T$  on  $X$  is defined dually. A left action of  $T$  on  $X$  can be equivalently given by a monoid homomorphism  $\varphi : t \mapsto \varphi_t$ ,  $\varphi_t(x) = t * x$ , from  $T$  to the full transformation monoid  $\mathcal{T}(X)$ . A right action can be given by an anti-homomorphism in a similar way.

A map  $\pi : X \rightarrow Y$  between posets is called *order-preserving* if  $x \leq y$  implies  $\pi(x) \leq \pi(y)$  for all  $x, y \in X$ . It is called an *order-embedding* if  $x \leq y$  holds if and only if  $\pi(x) \leq \pi(y)$  for all  $x, y \in X$ . An order-embedding is necessarily an injective map. A bijective order-embedding is called an *order-isomorphism*. If  $X = Y$  then order-isomorphisms are referred

to as *order-automorphisms*. A left action of  $T$  on a poset  $X$  given by a homomorphism  $\varphi : T \rightarrow \mathcal{T}(X)$  is called *order-preserving* (an action *by order-embeddings* or *by order-automorphisms*) if for each  $t \in T$  the transformation  $\varphi_t$  is order-preserving (resp. an order-embedding or an order-automorphism).

Let  $\cdot$  and  $*$  be left actions of monoids  $T$  and  $T'$  on posets  $X$  and  $X'$ , respectively. These actions are called *isomorphic* if there are a monoid isomorphism  $\alpha : T \rightarrow T'$  and an order-isomorphism  $\tau : X \rightarrow X'$  such that  $\alpha(t) * (\tau(x)) = \tau(t \cdot x)$  for all  $t \in T$  and  $x \in X$ .

These concepts can be readily adapted to right actions.

**2.4.  $W$ -products.** We now recall the (left hand version of the) construction of a  $W$ -product of a semilattice by a monoid, which is a generalization of the construction of a semidirect product of a semilattice by a group and shares several of its important properties [4, 19, 20].

Let  $T$  be a monoid and  $*$  be a left action of  $T$  on a semilattice  $Y$  by order-embeddings. Assume that the ranges of actions of the elements of  $T$  are order ideals of  $Y$ , that is if  $x \leq t * y$  for  $t \in T$  and  $x, y \in Y$  then there is  $z \in Y$  such that  $x = t * z$ . Set

$$W(T, Y) = \{(t * y, t) \in Y \times T : y \in Y, t \in T\}$$

and define the multiplication and the unary operations  $\lambda$  and  $\rho$  on  $W(T, Y)$  by

$$\begin{aligned} (t * y, t)(s * x, s) &= (t * y \wedge (ts) * x, ts); \\ \lambda(t * y, t) &= (y, 1); \quad \rho(t * y, t) = (t * y, 1). \end{aligned}$$

Then  $W(T, Y)$  is a proper restriction semigroup with respect to the semilattice

$$\overline{Y} = \{(y, 1) : y \in Y\}$$

which is isomorphic to  $Y$ . Furthermore,  $W(T, Y)/\sigma \simeq T$  via the map  $(t * y, t) \mapsto t$ . The semigroup  $W(T, Y)$  is a monoid if and only if  $Y$  has an identity.

**2.5. Free restriction monoids and semigroups.** Free restriction monoids and semigroups will appear throughout the paper as examples illustrating our constructions. We briefly recall their structure, and refer the reader to [3, 19, 20] for further details. Let  $Z$  be a set,  $Z^*$  the free monoid over  $Z$  and let  $1$  the empty word. By  $F\mathcal{G}(Z)$  we denote the free group over  $Z$ . The elements of  $F\mathcal{G}(Z)$  are reduced words over  $Z \cup Z^{-1}$ . If  $u, v \in F\mathcal{G}(Z)$ , their product  $\text{red}(uv)$  is the reduced word equivalent to the word  $uv$  obtained by the concatenation of  $u$  and  $v$ . The set  $F\mathcal{G}(Z)$  is partially ordered by the *prefix order*  $\leq_p$ . Let  $\mathcal{Y}'$  be the set of all finite order ideals of  $(F\mathcal{G}(Z), \leq_p)$ , and let  $\mathcal{Y} = \mathcal{Y}' \setminus \{1\}$ . The group  $F\mathcal{G}(Z)$  acts on the left of its powerset by

$$v * S = \{\text{red}(vs) : s \in S\}.$$

Let

$$\begin{aligned} \mathcal{X}' &= F\mathcal{G}(Z) * \mathcal{Y}', & \mathcal{X} &= F\mathcal{G}(Z) * \mathcal{Y}, \\ \mathcal{Q}' &= Z^* * \mathcal{Y}', & \mathcal{Q} &= Z^* * \mathcal{Y}. \end{aligned}$$

With respect to the reverse inclusion order  $\mathcal{X}'$ ,  $\mathcal{X}$ ,  $\mathcal{Q}'$  and  $\mathcal{Q}$  are semilattices. Clearly,  $\mathcal{X}'$  and  $\mathcal{X}$  are invariant under the action of  $F\mathcal{G}(Z)$  and  $\mathcal{Q}'$ ,  $\mathcal{Q}$  are invariant under the action of

$Z^*$ . There is also a right action  $\bullet$  of  $F\mathcal{G}(Z)$  on its powerset given by  $S \bullet v = v^{-1} * S$  and both  $\mathcal{X}'$  and  $\mathcal{X}$  are invariant under this action. However, neither  $\mathcal{Q}'$  nor  $\mathcal{Q}$  is invariant under  $\bullet$ . The left action  $*$  of  $Z^*$  on  $\mathcal{Q}'$  satisfies the requirements for forming of the  $W$ -product  $W(Z^*, \mathcal{Q}')$ . The set

$$F_W\mathcal{RM}(Z) = \{(t * y, t) \in W(Z^*, \mathcal{Q}') : y \in \mathcal{Y}' \text{ and } t * y \in \mathcal{Y}'\}$$

forms a  $(2, 1, 1)$ -subalgebra of  $W(Z^*, \mathcal{Q}')$  which is isomorphic to the free restriction monoid  $F\mathcal{RM}(Z)$  over  $Z$ . We refer to it as the *Szendrei's model* of the free restriction monoid. Note that

$$P(F_W\mathcal{RM}(Z)) = \{(y, 1) : y \in \mathcal{Y}'\}$$

is a semilattice isomorphic to  $\mathcal{Y}'$  via the map  $(y, 1) \mapsto y$ ;  $(t * y, t) \sigma (s * x, s)$  if and only if  $t = s$  and  $W(Z^*, \mathcal{Q}')/\sigma \simeq Z^*$  via the map  $[t * y, t] \mapsto t$ .

Similarly, we can form the  $W$ -product  $W(Z^*, \mathcal{Q})$  and construct the Szendrei's model of the free restriction semigroup  $F\mathcal{RS}(Z)$  over  $Z$  as its  $(2, 1, 1)$ -subalgebra  $F_W\mathcal{RS}(Z)$  by replacing  $\mathcal{Q}'$  by  $\mathcal{Q}$  and  $\mathcal{Y}'$  by  $\mathcal{Y}$  in the definition of  $F_W\mathcal{RM}(Z)$ .

### 3. CLASSES OF PROPER RESTRICTION SEMIGROUPS

**3.1. Partial actions and Cornock-Gould structure theorem.** Let  $T$  be a monoid with the identity element 1 and  $X$  a semilattice. A *left partial action* of  $T$  on  $X$  is a partial map  $T \times X \rightarrow X$ ,  $(t, x) \mapsto t \cdot x$ , such that

(LP1) for all  $x \in X$ :  $1 \cdot x$  is defined and  $1 \cdot x = x$ .

(LP2) for all  $s, t \in T$  and  $x \in X$ : if  $t \cdot x$  is defined and  $s \cdot (t \cdot x)$  is defined then  $(st) \cdot x$  is defined and  $s \cdot (t \cdot x) = (st) \cdot x$ .

A *right partial action* of  $T$  on  $X$  is defined dually.

A left partial action of a monoid  $T$  on a set  $X$  can be looked at as a dual prehomomorphism from  $T$  to the partial transformation monoid  $\mathcal{PT}(X)$ . Recall that a map  $\varphi : T \rightarrow \mathcal{PT}(X)$ ,  $t \mapsto \varphi_t$ , is called a *dual prehomomorphism* if  $\varphi_1$  is the identity transformation and  $\varphi_{st}$  is an extension of  $\varphi_s \varphi_t$ , that is, if  $x \in \text{dom}(\varphi_s \varphi_t)$  then  $x \in \text{dom}(\varphi_{st})$  and  $\varphi_s \varphi_t(x) = \varphi_{st}(x)$ . *Order-preserving* partial actions, partial actions by *order-isomorphisms* and *isomorphic partial actions* on posets are defined similarly as analogous notions for actions.

Let  $Y$  be a semilattice and  $T$  a monoid acting on  $Y$  partially on the left via  $\cdot$  and  $\varphi : T \rightarrow \mathcal{PT}(Y)$  be the corresponding dual prehomomorphism. Assume that for every  $t \in T$  the map  $\varphi_t$  satisfies the following axioms:

- (A)  $\text{dom}(\varphi_t)$  and  $\text{ran}(\varphi_t)$  are order ideals of  $Y$ .
- (B)  $\varphi_t : \text{dom}(\varphi_t) \rightarrow \text{ran}(\varphi_t)$  is an order-isomorphism.
- (C)  $\text{dom}(\varphi_t) \neq \emptyset$ .

Axiom (B) says that the image of  $\varphi$  is contained in the symmetric inverse monoid  $\mathcal{I}(Y)$  consisting of all partial injective maps between subsets of  $Y$ .

**Lemma 3.** *The assignment  $t \rightarrow \varphi_t^{-1}$  defines a right partial action,  $\circ$ , of  $T$  on  $Y$  and its defining dual antiprehomomorphism  $\psi$ , given by  $\psi_t = \varphi_t^{-1}$ , satisfies axioms (A), (B) and (C).*

*Proof.* It is immediate that  $y \circ 1 = y$  for all  $y \in Y$ . Assume that  $y \circ s$  and  $(y \circ s) \circ t$  are defined. Let  $x = y \circ s$  and  $z = x \circ t$ . Then  $y = s \cdot x$  and  $x = t \cdot z$  whence  $y = s \cdot (t \cdot z) = st \cdot z$  since  $\cdot$  is a partial action. Thus  $y \circ st$  is defined and  $y \circ st = z$ . Axioms (A), (B), (C) for  $\psi$  are straightforward to verify.  $\square$

We say that the partial actions  $\cdot$  and  $\circ$  are *reverse* to each other. It is immediate that for every  $t \in T$  and  $y \in Y$ :

$$\text{if } t \cdot y \text{ is defined then } (t \cdot y) \circ t \text{ is defined and } (t \cdot y) \circ t = y;$$

$$\text{if } y \circ t \text{ is defined then } t \cdot (y \circ t) \text{ is defined and } t \cdot (y \circ t) = y.$$

Given a left partial action  $\cdot$  of  $T$  on  $Y$  such that axioms (A), (B) and (C) hold set

$$M(T, Y) = \{(y, t) \in Y \times T : y \circ t \text{ is defined}\}$$

and define the multiplication on  $M(T, Y)$  by

$$(x, s)(y, t) = (s \cdot ((x \circ s) \wedge y), st).$$

With respect to this multiplication,  $M(T, Y)$  is a semigroup. Furthermore, elements of the form  $(y, 1)$ , where  $y$  runs through  $Y$ , form a subsemilattice  $\overline{Y}$  isomorphic to  $Y$ . It is shown in [1] that  $M(T, Y)$  is a proper restriction semigroup with respect to  $\overline{Y}$  where the functions  $\lambda$  and  $\rho$  are given by  $\lambda(y, t) = (y \circ t, 1)$ ,  $\rho(y, t) = (y, 1)$ .

Conversely, let  $S$  be a proper restriction semigroup. There is a left partial action  $\cdot$  of  $T = S/\sigma$  on  $E = P(S)$  which is given for  $t \in S/\sigma$  and  $e \in E$  by:

$$(3.1) \quad t \cdot e \text{ is defined if and only if there exists } a \in t \text{ such that } \lambda(a) \geq e,$$

in which case  $t \cdot e = \rho(ae)$ . Notice that if  $\lambda(a) \geq e$  then  $\lambda(ae) = e$  and so there exists  $a' = ae \in t$  with  $\lambda(a') = e$ . We will use this fact in the sequel without further mention. The partial action  $\cdot$  satisfies axioms (A), (B), (C) and the following structure theorem holds:

**Theorem 4** (Cornock and Gould [1]). *Any proper restriction semigroup  $S$  with respect to  $E$  is isomorphic to  $M(S/\sigma, E)$ .*

Throughout the paper, we refer to the left partial action  $\cdot$  and its reverse right partial action  $\circ$  as the partial actions *underlying*  $S$ .

**3.2. Partial actions and  $W$ -products.** Let  $W(T, Y)$  be a  $W$ -product of a semilattice  $Y$  by a monoid  $T$  and  $*$  the left action of  $T$  on  $Y$  defining it. It is immediate that axioms (A), (B) and (C) are satisfied by  $*$  so that the semigroup  $M(T, Y)$  may be formed. By construction, the sets  $M(T, Y)$  and  $W(T, Y)$  coincide and so do the unary operations  $\lambda$  and  $\rho$  on these sets. Furthermore, a direct calculation (or an application of Lemma 10) shows that the products on  $M(T, Y)$  and  $W(T, Y)$  coincide, too. It follows that  $M(T, Y) = W(T, Y)$  as  $(2, 1, 1)$ -algebras. Conversely, let  $*$  be a left action of  $T$  on  $Y$  which satisfies

axioms (A), (B) and (C). Then  $*$  satisfies the requirements needed to form the  $W$ -product  $W(T, Y)$ . We obtain the following statement.

**Proposition 5.**  *$W$ -products of semilattices by monoids (defined by left actions) are precisely the proper restriction semigroups whose underlying left partial action is an action.*

**3.3.  $F$ -restriction monoids.** By analogy with inverse semigroups, we call a restriction semigroup an  $F$ -restriction semigroup if every  $\sigma$ -class has the maximum element.

**Lemma 6.** *Let  $S$  be an  $F$ -restriction semigroup. Then  $S$  is proper. Consequently,  $S$  is necessarily a monoid with the identity being the maximum projection.*

*Proof.* For  $a \in S$  let  $m(a)$  be the maximum element in the  $\sigma$ -class of  $a$ . Assume that  $a \sigma b$ . Then  $m(a) = m(b)$ . Since  $a, b \leq m(a)$ , we get  $a \sim b$  by Lemma 1(1). Hence  $\sigma \subseteq \sim$ . Since the converse inclusion is clear, we obtain  $\sigma = \sim$  and thus  $S$  is proper by Lemma 2.  $\square$

Recall that the *Munn semigroup*  $T_Y$  of a semilattice  $Y$  is the semigroup of all order-isomorphism between principal order ideals of  $Y$  under composition. This is an inverse semigroup contained in  $\mathcal{I}(Y)$ .

**Lemma 7.** *Let  $T$  be a monoid acting partially on the left of a semilattice  $Y$  so that the semigroup  $M(T, Y)$  can be formed and let  $\varphi : T \rightarrow \mathcal{I}(Y)$  be the corresponding dual prehomomorphism. The following statements are equivalent:*

- (1)  $M(T, Y)$  is an  $F$ -restriction monoid;
- (2)  $\text{dom}(\varphi_t)$  is a principal order ideal for every  $t \in T$ ;
- (3) The image of  $\varphi$  is contained in the Munn semigroup  $T_Y$  of  $Y$ .

*Proof.* Assume that the semigroup  $M(T, Y)$  is  $F$ -restriction. Since  $(x, s) \sigma (y, t)$  holds if and only if  $s = t$  and  $(x, s) \geq (y, s)$  holds if and only if  $x \geq y$ , we conclude that for every  $t \in T$  there is a maximum element  $y \in Y$  such that  $(y, t) \in M(T, Y)$ . That is,  $y$  is the maximum element for which  $y \circ t$  is defined or, equivalently,  $y$  is the maximum element of  $\text{ran}(\varphi_t)$ . Hence  $y \circ t$  is the maximum element of  $\text{dom}(\varphi_t)$ . The converse implication is proved by reversing the arguments. The equivalence of the second and the third assertions is immediate.  $\square$

**Corollary 8.** *A  $W$ -product  $W(T, Y)$  is  $F$ -restriction if and only if  $Y$  has an identity if and only if  $W(T, Y)$  is a monoid.*

**3.4. Strong partial actions.** A left partial action  $\cdot$  of a monoid  $T$  on a set  $Y$  is called *strong* if the following requirement holds:

- (S) for all  $s, t \in T$  and  $y \in Y$ : if  $t \cdot y$  and  $(st) \cdot y$  are defined then  $s \cdot (t \cdot y)$  is defined.

If the above condition is met we have  $s \cdot (t \cdot y) = (st) \cdot y$ . A *strong right partial action* is defined dually. Strong partial actions of monoids were first considered in [16] where Condition (S) is a part of the definition of a partial action, and the term strong is not used. They were then studied [6, 8]. If the monoid  $T$  is a group, its left partial action, as defined by (LP1) and (LP2), is a wider notion than the usual partial action of a group [2, 11], since the latter has to satisfy an additional requirement: for any  $g \in G$  and  $x \in X$



if  $g \cdot x$  is defined then also  $g^{-1} \cdot (g \cdot x)$  is defined and  $g \cdot (g^{-1} \cdot x) = x$ . A left partial action of a group, as defined by (LP1) and (LP2), is strong if and only if it is a usual left partial action.

A *globalization* of a left partial action  $\cdot$  of a monoid  $T$  on a set  $Y$  consists of (i) a left partial action  $\hat{\cdot}$  of  $T$  on a set  $\bar{Y}$  such that  $\hat{\cdot}$  is isomorphic to  $\cdot$  and (ii) a left action  $*$  of  $T$  on a superset  $X \supseteq \bar{Y}$ , such that for every  $y \in \bar{Y}$ :  $t \hat{\cdot} y = t * y$  whenever  $t \hat{\cdot} y$  is defined. Strong left partial actions are precisely the left partial actions that can be globalized (easy to verify or see [8]). A similar definition and remark apply to right partial actions.

A dual concept to strongness, antistrongness, arises for partial actions by injective maps. Namely, if  $\cdot$  is such a left partial action and  $\varphi : T \rightarrow \mathcal{I}(Y)$ ,  $t \mapsto \varphi_t$ , the corresponding dual prehomomorphism then the assignment  $t \mapsto \varphi_t^{-1}$  defines a dual antiprehomomorphism and thus a right partial action,  $\circ$ . It is natural to call  $\cdot$  *antistrong* if  $\circ$  is strong. If  $T$  is a group then its (usual) left partial action is strong if and only if it is antistrong, but this is not the case for monoid actions in general.

**3.5. Strong partial actions and a condition from [1].** Let  $S$  be a proper restriction semigroup. We argue that strongness of the underlying partial actions  $\cdot$  and  $\circ$  of  $S/\sigma$  on  $E = P(S)$  is equivalent to the following conditions  $(EP)^r$  and  $(EP)^l$  which arise in [1].

$(EP)^r$  for all  $s, t, u \in S$ : if  $s \sigma tu$  then there exists  $v \in S$  with  $\rho(t)s = tv$  and  $u \sigma v$ .

The condition  $(EP)^l$  is defined dually.

**Proposition 9.** *Let  $S$  be a proper restriction semigroup. Then*

- (1)  *$S$  satisfies condition  $(EP)^r$  if and only if  $\circ$  is strong.*
- (2)  *$S$  satisfies condition  $(EP)^l$  if and only if  $\cdot$  is strong.*

*Proof.* Suppose that  $(EP)^r$  holds and show that  $\circ$  is strong. Let  $p, q \in S/\sigma$  and  $x \in E$  be such that  $x \circ (pq)$  and  $x \circ p$  are defined. Let further  $y = x \circ p$  and  $s \in pq$  and  $t \in p$  be such that  $x = \rho(s) = \rho(t)$  and  $y = \lambda(t)$ . Take any element  $u \in q$ . By  $(EP)^r$  there is  $v \in q$  such that  $\rho(t)s = tv$ . Since  $\rho(t) = \rho(s)$  this yields  $s = tv$ . Now, letting  $e = \rho(v)$ , we notice that  $s = tv = tev = \rho(te)tv = \rho(te)s$  which implies  $\rho(te) \geq \rho(s) = \rho(t)$ . Hence  $t \geq te = \rho(te)t \geq \rho(t)t = t$ , so that  $t = te$  and consequently  $e \geq \lambda(t) = y$ . Thus  $y \circ q$  is defined and therefore  $\circ$  is strong.

Conversely, suppose that the partial action  $\circ$  is strong and show that  $(EP)^r$  holds. For  $s \in S$  let  $[s]$  be its  $\sigma$ -class. Assume that  $s, t, u \in S$  are such that  $s \sigma tu$ . Then  $\rho(s)\rho(t) \circ [tu]$  is defined. Since  $\rho(s)\rho(t) \circ [t]$  is obviously defined, too, and since  $\circ$  is strong, it follows that  $(\rho(s)\rho(t) \circ [t]) \circ [u]$  is defined. Thus there exists  $v' \in [u]$  satisfying  $\rho(v') \geq \rho(s)\rho(t) \circ [t] = \lambda(\rho(s)t)$ . Then

$$\rho(\rho(s)tv') = \rho(\rho(s)t\rho(v')) = \rho(\rho(s)t) = \rho(s)\rho(t) = \rho(\rho(t)s).$$

Since also  $\rho(s)tv' \sigma tu \sigma \rho(t)s$  we obtain  $\rho(t)s = \rho(s)tv' = t\lambda(\rho(s)t)v'$ . Set  $v = \lambda(\rho(s)t)v'$ . Then  $v \in [u]$  and  $\rho(t)s = tv$ , so that  $(EP)^r$  holds.

The second statement is proved similarly. □

We call a proper restriction semigroup  $S$  *left extra proper* (*right extra proper*) if the underlying left partial action  $\cdot$  (respectively, the underlying right partial action  $\circ$ ) is strong. We call  $S$  *extra proper* if both  $\cdot$  and  $\circ$  are strong. Proposition 9 shows that this terminology agrees with that proposed in [1].

**3.6. Strong partial actions and FA-monoids from [3].** It is easy to verify that FA-monoids considered in [3, Section 9] are precisely ample extra proper  $F$ -restriction monoids.

**3.7. Partial actions restricting global actions.** Let  $*$  be an order-preserving left action of a monoid  $T$  on a poset  $X$  and  $Y$  be a subset of  $X$  which is an order ideal of  $X$  and a meet semilattice under the induced order. Furthermore, we assume that the induced partial action of  $T$  on  $Y$  satisfies axioms (A), (B) and (C), so that the semigroup  $M(T, Y)$  can be formed. We additionally assume that for every  $t \in T$  and  $x \in X$

$$(3.2) \quad \text{if } x \leq t * y \text{ where } y \in Y \text{ then } x = t * z \text{ for some } z \leq y.$$

**Lemma 10.** *For any  $x, y \in Y$  and  $s \in S$  such that  $x \circ s$  is defined the meet  $x \wedge s * y$  exists in  $X$  and*

$$x \wedge s * y = s \cdot ((x \circ s) \wedge y).$$

where  $\circ$  is the right partial action reverse to  $\cdot$ .

*Proof.* Since  $Y$  is a semilattice and both  $x \circ s$  and  $y$  belong to  $Y$ , the meet  $(x \circ s) \wedge y$  exists in  $Y$ . Let  $p = (x \circ s) \wedge y$ . Since  $p \leq x \circ s$  and  $s \cdot (x \circ s)$  is defined,  $s \cdot p$  is defined and  $s \cdot p \leq x$ . From  $p \leq y$  we have  $s \cdot p = s * p \leq s * y$  and thus  $s \cdot p$  is a lower bound for  $x$  and  $s * y$ . Assume that  $q \leq x, s * y$ . Since  $q \leq s * y$ , by (3.2) there is some  $z \leq y$  such that  $q = s * z$ . But  $z, q \in Y$  so that  $q = s \cdot z$  and  $z = q \circ s$ . Now,  $q \leq x$  implies  $q \circ s \leq x \circ s$  and so  $q \circ s \leq (x \circ s) \wedge y = p$ . This yields  $q \leq s \cdot p$ . We have proved that  $x \wedge (s * y)$  exists in  $X$  and equals  $s \cdot p = s \cdot ((x \circ s) \wedge y)$ , as required.  $\square$

**3.8. Ultra proper restriction semigroups.** A left partial action  $\cdot$  of a monoid  $T$  on a set  $Y$  will be called a *partially defined action*<sup>1</sup> if for all  $s, t \in T$  and  $x \in Y$  the following condition is met:

(PDA)  $(st) \cdot x$  is defined if and only if  $t \cdot x$  and  $s \cdot (t \cdot x)$  are defined.

Clearly, the above condition holds if and only if the dual prehomomorphism  $\varphi$  corresponding to  $\cdot$  is in fact a homomorphism. We call a proper restriction semigroup *ultra proper* if its underlying left partial action  $\cdot$  is a partially defined action.

**Lemma 11.** *A proper restriction semigroup  $S$  is ultra proper if and only if its underlying right partial action  $\circ$  is a partially defined action.*

*Proof.* Assume that  $\cdot$  is a partially defined action and show that so is  $\circ$ . Let  $x \in P(S)$  and  $s, t \in S/\sigma$  be such that that  $x \circ st$  is defined and put  $y = x \circ st$ . Then  $x = st \cdot y$ . Since  $\cdot$  is a partially defined action,  $t \cdot y$  and  $s \cdot (t \cdot y)$  are defined. Let  $z = t \cdot y$  and  $u = s \cdot z$ . Then  $y = z \circ t$  and  $z = u \circ s$  whence  $y = (u \circ s) \circ t$ . Thus  $\circ$  is a partially defined action. The ‘if’ part follows by symmetry.  $\square$

<sup>1</sup>The term *partial action* would be more appropriate here, but in this paper it has a different meaning

Hence a left partially defined action of a monoid on a semilattice which satisfies axioms (A), (B), (C) is necessarily both strong and antistrong and we have the following inclusions of classes of restriction semigroups:

$$\text{Proper} \supset \text{Extra proper} \supset \text{Ultra proper}$$

**Example 12.** By Proposition 5  $W$ -products of semilattices by monoids are ultra proper.

**Example 13.** Observe that, somewhat surprisingly, the free restriction monoid and the free restriction semigroup over a set  $Z$  are ultra proper. We verify this, for example, for the free restriction monoid  $F\mathcal{RM}(Z)$ . The underlying left partial action of  $Z^*$  on  $\mathcal{Y}'$  is given by:  $t \cdot A$  is defined if and only if  $t^{-1} \in A$  and in the latter case  $t \cdot A = t * A$ . Assume that  $(st) \cdot A$  is defined. Then  $t^{-1}s^{-1} \in A$ . Since  $A$  is prefix-closed,  $t^{-1} \in A$ . Hence  $t \cdot A$  is defined. Further,  $t^{-1}s^{-1} \in A$  implies  $s^{-1} \in t * A = t \cdot A$  so that  $s \cdot (t \cdot A)$  is defined.

**3.9. Ultra  $F$ -restriction monoids.** We call a restriction monoid  $S$  *ultra  $F$ -restriction* if it is ultra proper and  $F$ -restriction. This means that the left partial action  $\cdot$  underlying  $S$  is a partially defined action and  $\text{dom}(\varphi_t)$  (and then also  $\text{ran}(\varphi_t)$ ) is a principal order ideal for every  $t \in S/\sigma$  where  $\varphi: S/\sigma \rightarrow \mathcal{I}(P(S))$  is the homomorphism corresponding to  $\cdot$ . In other words,  $\varphi$  is a monoid homomorphism from  $S/\sigma$  to the Munn monoid  $T_{P(S)}$  of the semilattice  $P(S)$ .

**Example 14.** The free restriction monoid  $F\mathcal{RM}(Z)$  is ultra  $F$ -restriction. This follows from Example 13 and the fact that  $\text{dom}(\varphi_t)$  is a principal order ideal of  $\mathcal{Y}'$  (recall that the order on  $\mathcal{Y}'$  is the reverse inclusion) generated by  $\{1, t_1, t_1t_2, \dots, t_1 \cdots t_n\}$  where  $t^{-1} = t_1 \cdots t_n$  and  $t_i \in Z$  for all  $i$ . Note that  $F\mathcal{RS}(Z)$  is not  $F$ -restriction since  $\text{dom}(\varphi_1) = \mathcal{Y}$  is not a principal order ideal.

**3.10. An ultra  $F$ -restriction cover of a restriction monoid.** Let  $S$  be a restriction monoid and  $A$  be its generating set as a  $(2, 1, 1, 0)$ -algebra. Let  $T = A^*$  be the free monoid generated by  $A$  (in the usual monoid signature  $(2, 0)$ ). Put  $E = P(S)$  and for  $v \in T$  let  $\bar{v}$  be the value of  $v$  in  $S$ . For  $v \in T$  and  $e \in E$  we set

$$v \cdot e \text{ is defined if and only if } \lambda(\bar{v}) \geq e.$$

If  $v \cdot e$  is defined we put  $v \cdot e = \rho(\bar{v}e)$ .

**Lemma 15.**

- (1) *The partial map  $\cdot$  is a partially defined action of  $T$  on  $E$  and the monoid  $M(T, E)$  is ultra  $F$ -restriction (note that  $M(T, E)$  is even ample since  $T$  is cancellative).*
- (2) *The map  $M(T, E) \rightarrow S$  given by  $(e, v) \mapsto e\bar{v}$  is a surjective projection separating  $(2, 1, 1, 0)$ -morphisms.*

*Proof.* Assume that  $vw \cdot e$  is defined and show that  $w \cdot e$  and  $v \cdot (w \cdot e)$  are defined. By assumption  $\lambda(\bar{v}\bar{w}) \geq e$ . Therefore,

$$\lambda(\bar{w}) \geq \lambda(\lambda(\bar{v})\bar{w}) = \lambda(\bar{v}\bar{w}) = \lambda(\bar{v}\bar{w}) \geq e.$$

Thus  $w \cdot e$  is defined. To show that  $v \cdot (w \cdot e)$  is defined we verify that  $\lambda(\bar{v}) \geq \rho(\bar{w}e)$ . The latter is equivalent to  $\bar{w}e = \lambda(\bar{v})\bar{w}e$ . We have

$$\lambda(\bar{v})\bar{w}e = \bar{w}\lambda(\lambda(\bar{v})\bar{w})e = \bar{w}\lambda(\bar{v}\bar{w})e = \bar{w}e,$$

as required.

Assume now that  $w \cdot e$  and  $v \cdot (w \cdot e)$  are defined and show that  $wv \cdot e$  is defined. By assumption  $\lambda(\bar{w}) \geq e$  and  $\lambda(\bar{v}) \geq \rho(\bar{w}e)$ . Then we have

$$\lambda(\bar{v}\bar{w}) = \lambda(\lambda(\bar{v})\bar{w}) \geq \lambda(\rho(\bar{w}e)\bar{w}) = \lambda(\bar{w}e) = e.$$

If both  $w \cdot e$  and  $v \cdot (w \cdot e)$  are defined then

$$v \cdot (w \cdot e) = \rho(\bar{v}\rho(\bar{w}e)) = \rho(\bar{v}\bar{w}e) = vw \cdot e.$$

The remaining axioms of a partial action, as well as that each  $\sigma$ -class of  $M(T, E)$  has the maximum element, are easy to verify.

For the second item, we first show that the given assignment preserves the multiplication. Let  $\circ$  be the right partially defined action reverse to  $\cdot$ . Then  $e \circ v$  is defined if and only if  $\rho(\bar{v}) \geq e$  and whenever defined it equals  $\lambda(e\bar{v})$ . Using this, we calculate, for any  $(e, v), (f, u) \in M(T, E)$ :

$$\begin{aligned} (e, v)(f, u) &= (v \cdot ((e \circ v) \wedge f), vu) = (\rho(\bar{v}\lambda(e\bar{v})f), vu) = (\rho(e\bar{v}f), vu) \mapsto \rho(e\bar{v}f)\bar{v}u; \\ e\bar{v}f\bar{u} &= e\rho(\bar{v}f)\bar{v}\bar{u} = \rho(e\rho(\bar{v}f))\bar{v}\bar{u} = \rho(e\bar{v}f)\bar{v}\bar{u}. \end{aligned}$$

So preservation of the multiplication is verified. It is easy to see that  $\lambda$ ,  $\rho$  and identity are preserved, too. That the assignment is projection separating is immediate.  $\square$

**3.11. An ultra proper cover of a restriction semigroup.** Let  $S$  be a restriction semigroup and let  $A$  be its generating set as a  $(2, 1, 1)$ -algebra. Let  $T = A^*$  be the free monoid where the empty word will be denoted by  $\epsilon$ . If  $v \in A^+ = A^* \setminus \{\epsilon\}$  by  $\bar{v}$  we denote the value of  $v$  in  $S$ . We also let  $E = P(S)$ . For  $v \in T$  and  $e \in E$  we set:

$$v \cdot e \text{ is defined if and only if } v = \epsilon \text{ or, otherwise, } \lambda(\bar{v}) \geq e.$$

If  $v \cdot e$  is defined we put

$$v \cdot e = \begin{cases} e, & \text{if } v = \epsilon; \\ \rho(\bar{v}e), & \text{otherwise.} \end{cases}$$

Slightly modifying the arguments of the proof of Lemma 15 it can be shown that  $\cdot$  is a partially defined action such that the semigroup  $M(T, E)$  can be formed and is ultra proper. Furthermore, the map  $\varphi : M(T, E) \rightarrow S$  given by  $(e, \epsilon) \mapsto e$  and  $(e, v) \mapsto e\bar{v}$ , if  $v \neq \epsilon$  is a surjective projection separating  $(2, 1, 1)$ -morphism.

The monoid  $S^1 = S \cup \{1\}$  where  $1 \notin S$  becomes a restriction monoid with respect to  $P(S^1) = E^1$  if we extend the operations  $\lambda$  and  $\rho$  to  $S^1$  by setting  $\lambda(1) = \rho(1) = 1$ . We refer to  $S^1$  as the restriction monoid obtained from the restriction semigroup  $S$  by *adjoining an identity element*. Clearly, the inclusion map  $S \rightarrow S^1$  is a  $(2, 1, 1)$ -morphism. The set  $A$  is a generating set of  $S^1$  as a  $(2, 1, 1, 0)$ -algebra and applying the construction of Section 3.10 we can construct the monoid  $M(T, E^1)$  and the cover  $M(T, E^1) \rightarrow S^1$ . It is now straightforward to verify that  $M(T, E^1) = M(T, E) \cup (\epsilon, 1)$  as sets and, moreover,

$M(T, E^1)$  is a restriction monoid obtained from the restriction semigroup  $M(T, E)$  by adjoining an identity element.

**3.12. Ultra  $F$ -restriction monoids are the monoids  $Y *_m T$  from [3].** In this section we show that a monoid is ultra  $F$ -restriction if and only if it is  $(2, 1, 1, 0)$ -isomorphic to a monoid  $Y *_m T$  considered in [3, 7]. We now recall its definition.

There is a monoid  $T$  and a semilattice  $Y$  with identity,  $\epsilon$ . Furthermore, there are a left action  $*$  and a right action  $\bullet$  of  $T$  on  $Y$  such that for all  $t \in T$  and  $x, y \in Y$ :

$$(3.3) \quad t * (x \wedge y) = t * x \wedge t * y, \quad (x \wedge y) \bullet t = x \bullet t \wedge y \bullet t;$$

$$(3.4) \quad (t * x) \bullet t = (\epsilon \bullet t) \wedge x, \quad t * (x \bullet t) = x \wedge (t * \epsilon).$$

The actions  $*$  and  $\bullet$  are then said to form a *double action* of  $T$  on  $Y$ . Put

$$(3.5) \quad Y *_m T = \{(y, t) \in Y \times T : y \leq t * \epsilon\}$$

and define the multiplication on it by

$$(x, s)(y, t) = (x \wedge s * y, st).$$

Further, let

$$P(Y *_m T) = \{(y, t) \in Y *_m T : t = 1\}$$

and for every  $(y, t) \in Y *_m T$  put

$$\lambda(y, t) = (y \bullet t, 1), \quad \rho(y, t) = (y, 1).$$

**Proposition 16** ([3]).  *$Y *_m T$  is a proper restriction monoid with identity  $(\epsilon, 1)$ . Its semilattice of projections  $P(Y *_m T)$  is order isomorphic to  $Y$  via the map  $(y, 1) \mapsto y$ ;  $(y, t) \sigma (x, s)$  if and only if  $t = s$  and  $(Y *_m T)/\sigma \simeq T$  via the map  $[y, t] \mapsto t$ .*

It is natural to ask if the partial action underlying  $Y *_m T$  has some specific properties caused by the ‘symmetry’ of the double action defining it, a question which we now consider. Assume that  $*$  and  $\bullet$  form a double action of  $T$  on  $Y$ . Define the following partial map  $T \times Y \rightarrow Y$ :

$$(3.6) \quad t \cdot y \text{ is defined if and only if } y \leq \epsilon \bullet t \text{ in which case set } t \cdot y = t * y.$$

For  $y \in Y$  we set  $y^\downarrow = \{x \in Y : x \leq y\}$  to be the principal order ideal generated by  $y$ .

**Proposition 17.**

- (1) *The map  $\cdot$  is a partially defined action which satisfies (A), (B) and (C) so that we can form the semigroup  $M(T, Y)$ . Moreover,  $M(T, Y)$  is an ultra  $F$ -restriction monoid.*
- (2)  *$M(T, Y)$  and  $Y *_m T$  are equal as  $(2, 1, 1, 0)$ -algebras.*

*Proof.* Since  $\cdot$  is obtained by restricting a global action  $*$ , it is a strong partial action. To verify (B) we show that  $\varphi_t, x \mapsto t \cdot x$ , is an order-isomorphism between  $(\epsilon \bullet t)^\downarrow$  and  $(t * \epsilon)^\downarrow$ . If  $t \cdot x$  is defined then  $t \cdot x \leq t * \epsilon$  as  $x \leq \epsilon$  and  $*$  is order-preserving by (3.3). Assume  $x \leq t * \epsilon$ . Then  $x = t * (x \bullet t)$  by (3.4) and thus  $x = t \cdot (x \bullet t)$  since  $x \bullet t \leq \epsilon \bullet t$ . It follows that  $x \in \text{ran}(\varphi_t)$  and thus  $\text{ran}(\varphi_t) = (t * \epsilon)^\downarrow$ . Assume that  $t \cdot x \leq t \cdot y$ . As before, we have

$x = (t \cdot x) \bullet t$  and  $y = (t \cdot y) \bullet t$ . Hence  $x \leq y$  as  $\bullet$  is order-preserving by (3.3). Since  $\text{dom}(\varphi_t) = (\epsilon \bullet t)^\downarrow$  and  $\text{ran}(\varphi_t) = (t * \epsilon)^\downarrow$ , these are principal order ideals which subsumes (A). For  $t \in T$   $t \cdot (\epsilon \bullet t)$  is defined which verifies (C). We are left to verify that  $\cdot$  is a partially defined action. For this assume that  $ts \cdot y$  is defined and show that  $s \cdot y$  is defined. This is enough as we know that  $\cdot$  is strong. We have  $y \leq \epsilon \bullet ts = (\epsilon \bullet t) \bullet s \leq \epsilon \bullet s$ , and the first item is proved.

We now prove the second item. Let  $\circ$  be the right partial action reverse to  $\cdot$ . Observe that  $x \circ t$  is defined if and only if  $x \leq t * \epsilon$ , which implies that  $M(T, Y) = Y *_m T$  as sets. It is immediate that their unary operations and the identity elements coincide. We verify that the multiplications coincide, too. For this, we verify that  $x \wedge s * y = s * (x \bullet s \wedge y)$  whenever  $x \leq s * \epsilon$ :

$$\begin{aligned} s * (x \bullet s \wedge y) &= s * (x \bullet s) \wedge s * y && \text{(by (3.3))} \\ &= x \wedge s * \epsilon \wedge s * y && \text{(by (3.4))} \\ &= x \wedge s * y && \text{(since } x \leq s * \epsilon \text{).} \end{aligned}$$

□

In the opposite direction, let  $\cdot$  be a left partially defined action of a monoid  $T$  on a semilattice  $Y$  with identity satisfying axioms (A), (B), (C) such that the monoid  $M(T, Y)$  is ultra  $F$ -restriction. Our task is to construct a double action of  $T$  on  $Y$ . Let  $\circ$  be the right partially defined action reverse to  $\cdot$ . For  $t \in T$  we let  $d_t, r_t \in Y$  be the top elements of  $\text{dom}(\varphi_t)$  and  $\text{ran}(\varphi_t)$ , respectively.

For each  $t \in T$  and  $y \in Y$  we set

$$(3.7) \quad t * y = t \cdot (y \wedge d_t), \quad y \bullet t = (y \wedge r_t) \circ t.$$

**Proposition 18.**

- (1) *The maps  $*$  and  $\bullet$  form a double action of  $T$  on  $Y$ . Consequently, we can form the monoid  $Y *_m T$ .*
- (2)  *$Y *_m T$  and  $M(T, Y)$  are equal as  $(2, 1, 1, 0)$ -algebras.*

*Proof.* First, we verify that  $*$  is an action. Let  $t, s \in T$  and  $y \in Y$ . To show that  $ts * y = t * (s * y)$  we need to verify that

$$ts \cdot (y \wedge d_{ts}) = t \cdot (s \cdot (y \wedge d_s) \wedge d_t).$$

By (PDA)  $s \cdot (y \wedge d_{ts})$  and  $t \cdot (s \cdot (y \wedge d_{ts}))$  are defined and the left hand side of the above equality equals  $t \cdot (s \cdot (y \wedge d_{ts}))$ . So the needed equality is equivalent to the equality

$$s \cdot (y \wedge d_{ts}) = s \cdot (y \wedge d_s) \wedge d_t.$$

Denote the left hand side of the above equality by  $A$  and the right hand side by  $B$ . Since  $t \cdot A$  is defined, we have  $A \leq d_t$ . Further, since  $ts \cdot d_{ts}$  is defined and  $ts \cdot d_{ts} = t \cdot (s \cdot d_{ts})$  we obtain  $d_{ts} \leq d_s$ . Then  $A \leq s \cdot (y \wedge d_s)$  so that have proved that  $A \leq B$ .

To prove that  $B \leq A$ , we let  $x = B \circ s$ . Since  $B \leq d_t$  it follows that  $t \cdot (s \cdot x)$  is defined so that  $ts \cdot x$  is defined which implies  $x \leq d_{ts}$ . Since  $s \cdot x = B \leq s \cdot (y \wedge d_s)$  it follows that

$x \leq y \wedge d_s$ . Therefore  $x \leq d_{ts} \wedge y \wedge d_s = y \wedge d_{ts}$  whence  $B = s \cdot x \leq s \cdot (y \wedge d_{ts}) = A$ , as required. That  $\bullet$  is an action is established similarly.

Let  $t \in T$  and  $x, y \in Y$ . The first equality in (3.3) holds since

$$t * (x \wedge y) = t \cdot (x \wedge y \wedge d_t) = t \cdot ((x \wedge d_t) \wedge (y \wedge d_t)) = t \cdot (x \wedge d_t) \wedge t \cdot (y \wedge d_t) = t * x \wedge t * y.$$

The second equality is verified similarly.

For the first equality in (3.4) we calculate:

$$\begin{aligned} (t * x) \bullet t &= (r_t \wedge t \cdot (d_t \wedge x)) \circ t \\ &= (t \cdot (d_t \wedge x)) \circ t && (\text{since } t \cdot (d_t \wedge x) \leq r_t) \\ &= d_t \wedge x; \\ (\epsilon \bullet t) \wedge x &= (\epsilon \wedge r_t) \circ t \wedge x = d_t \wedge x. \end{aligned}$$

The second equality is verified similarly.

We now prove the second item. As in the previous lemma, it is immediate that  $Y *_m T$  and  $M(T, Y)$  are equal as sets and that their unary operations and identities coincide. We only verify that the multiplication in  $Y *_m T$  coincides with that in  $M(T, Y)$ . This reduces to verifying that  $x \wedge s \cdot (y \wedge d_s) = s \cdot (x \circ s \wedge y)$  whenever  $x \circ s$  is defined:

$$\begin{aligned} x \wedge s \cdot (y \wedge d_s) &= s \cdot (x \circ s) \wedge s \cdot (y \wedge d_s) \\ &= s \cdot (x \circ s \wedge y \wedge d_s) \\ &= s \cdot (x \circ s \wedge y) && (\text{since } x \circ s \leq d_s). \end{aligned}$$

□

**Remark 19.** Let  $M(T, Y)$  be an ultra  $F$ -restriction monoid defined by a left patially defined action  $\cdot$  of  $T$  on  $Y$  and  $\circ$  the rihgt partially defined action reverse to  $\cdot$ . Let further  $*$  and  $\bullet$  be the actions defined in (3.7). For every  $t \in T$  define the maps

$$\begin{aligned} \varphi_t : d_t^\downarrow &\rightarrow Y, \quad x \mapsto t \cdot x; \quad \psi_t : r_t^\downarrow \rightarrow Y, \quad x \mapsto x \circ t; \\ \tilde{\varphi}_t : Y &\rightarrow r_t^\downarrow, \quad x \mapsto t * x; \quad \tilde{\psi}_t : Y \rightarrow d_t^\downarrow, \quad x \mapsto x \bullet t. \end{aligned}$$

For every  $x \leq d_t$  and  $y \in Y$  we then have:

$$\varphi_t(x) \leq y \text{ if and only if } x \leq \tilde{\psi}_t(y).$$

Indeed,  $t \cdot x \leq y$  is equivalent to  $t \cdot x \leq y \wedge r_t$ , which is in turn equivalent to  $x \leq (y \wedge r_t) \circ t$ . Similarly for every  $x \leq r_t$  and  $y \in Y$ :

$$\psi_t(x) \leq y \text{ if and only if } x \leq \tilde{\varphi}_t(y).$$

This means that the map  $\varphi_t$  is a left adjoint to the map  $\tilde{\psi}_t$  and the map  $\psi_t$  is a left adjoint to the map  $\tilde{\varphi}_t$ .

**Remark 20.** Using the results of this section we can present the monoid  $M(T, E)$  from Section 3.10 in the form  $E *_m T$ . It is then easy to verify that it coincides (after a change in notation) with the covering monoid considered in the proof of Theorem 7.1 of [3].

## 4. GLOBALIZATION OF A STRONG PARTIAL ACTION

Let  $T$  be a monoid and  $\cdot$  a strong left partial action of  $T$  on a semilattice  $Y$  satisfying axioms (A), (B) and (C). In this section we construct a globalization  $*$  of  $\cdot$ . The construction is based on a combination of the ideas to be found in [16, 17]. Let  $\circ$  be the right partial action reverse to  $\cdot$ .

For  $(x, s), (y, t) \in Y \times T$  we set  $(x, s) \rightarrow (y, t)$  if there is  $p \in T$  such that  $s = tp$  and  $p \cdot x = y$ . So we have:

$$(x, tp) \rightarrow (p \cdot x, t); \quad (x, t) \leftarrow (x \circ p, tp)$$

whenever  $p \cdot x$  or  $x \circ p$  is defined.

Let  $\sim^2$  be the minimum equivalence relation on  $Y \times T$  which contains the relation  $\rightarrow$ . In other words,  $\sim$  is the transitive closure of  $\rightarrow \cup \leftarrow$ . For  $A, B \in (Y \times T)/\sim$  we define  $A \geq B$  if there are  $(x, s) \in A$  and  $(y, t) \in B$  such that  $x \geq y$ .

**Lemma 21.**

- (1) If  $A \geq B$  and  $(z, t) \in A$  then there is  $(u, t) \in B$  where  $z \geq u$ .
- (2) The relation  $\geq$  is a preorder on  $(Y \times T)/\sim$ .

*Proof.* To prove the first item, assume that  $A \geq B$ . Then there are  $(x, s) \in A$  and  $(y, s) \in B$  with  $x \geq y$ . Since  $(x, s) \sim (z, t)$ , there is a sequence  $(x, s) = (x_0, s_0), (x_1, s_1), \dots, (x_n, s_n) = (z, t)$  in  $Y \times T$  such that either  $(x_i, s_i) \rightarrow (x_{i+1}, s_{i+1})$  or  $(x_{i+1}, s_{i+1}) \rightarrow (x_i, s_i)$  for all admissible  $i$ . Assume that  $(x, s) \rightarrow (x_1, s_1)$ . Then there is a factorization  $s = s_1 q$  such that  $x_1 = q \cdot x$ . Then  $q \cdot y$  is defined and  $(y, s) \rightarrow (q \cdot y, s_1)$ . Put  $y_1 = q \cdot y$  and note that  $x_1 \geq y_1$ . Assume now that  $(x_1, s_1) \rightarrow (x, s)$ . Then there is a factorization  $s_1 = sp$  such that  $x_1 = x \circ p$  and  $s_1 = sp$ . Similarly as before, we get that  $y \circ p$  is defined and  $(y \circ p, sp) \rightarrow (y, s)$ . We put  $y_1 = y \circ p$  and note that  $y_1 \leq x_1$ . The statement now follows by induction.

For the second item note that reflexivity of  $\geq$  is obvious and transitivity follows from the previous item.  $\square$

**Lemma 22.** Let  $(x, s) \sim (y, t)$ . Then either both  $s \cdot x$  and  $t \cdot y$  are defined in which case  $s \cdot x = t \cdot y$ , or they are both undefined.

*Proof.* It is enough to consider only the case when  $(x, s) \rightarrow (y, t)$ , since the other case then holds by symmetry and the statement follows by induction. Rewriting  $(x, s) \rightarrow (y, t)$  as  $(x, tp) \rightarrow (p \cdot x, t)$ , we see that the claim holds since  $\cdot$  is strong.  $\square$

**Lemma 23.** Let  $A, B \in (Y \times T)/\sim$  be such that  $A \neq B$ ,  $A \leq B$  and  $B \leq A$ . If  $(x, s) \in A$  then  $s \cdot x$  is undefined. Consequently, no element of the form  $(x, 1)$  belongs to  $A$ .

*Proof.* Let  $(x, s) \in A$ . By Lemma 21 there are  $(y, s) \in B$  and  $(z, s) \in A$  such that  $x \geq z \geq y$ . Assume that  $s \cdot x$  is defined. Then by Lemma 22  $s \cdot y$  is defined and  $s \cdot x = s \cdot y$ . It follows that  $x = (s \cdot x) \circ s = (s \cdot y) \circ s = y$ , which is a contradiction.  $\square$

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<sup>2</sup>From now on  $\sim$  is used as is defined here and is no longer used to denote the compatibility relation



Let  $\approx$  be an equivalence on  $Y \times T$  given by  $(x, s) \approx (y, t)$  if and only if  $[x, s]_{\sim} \leq [y, t]_{\sim}$  and  $[y, t]_{\sim} \leq [x, s]_{\sim}$  where  $[x, s]_{\sim}$  denotes the  $\sim$ -class of  $(x, s)$ . Put  $X = (Y \times T)/\approx$ . It is partially ordered with the order induced by the preorder on  $(Y \times T)/\sim$ . By  $[x, s]$  we will denote the  $\approx$ -class of  $(x, s)$ . We let  $\overline{Y} = \{[y, 1] : y \in Y\}$ .

**Lemma 24.**

- (1) *The map  $\theta : y \mapsto [y, 1]$  is an order-isomorphism between  $Y$  and  $\overline{Y}$ .*
- (2)  *$\overline{Y}$  is an order ideal of  $X$ .*
- (3)  *$\overline{Y}$  is a meet semilattice under the induced order on  $\overline{Y}$  and  $Y$  is isomorphic to  $\overline{Y}$  as a meet semilattice via  $\theta$ .*

*Proof.* (1) Clearly,  $\theta$  is surjective. Let  $x, y \in Y$ . If  $x \leq y$  then  $[x, 1]_{\sim} \leq [y, 1]_{\sim}$  by the definition of the order in  $(Y \times T)/\sim$  and so  $[x, 1] \leq [y, 1]$ . Assume that  $[x, 1] \leq [y, 1]$ . Then  $[x, 1]_{\sim} \leq [y, 1]_{\sim}$  and by Lemma 21  $[x, 1]_{\sim} = [z, 1]_{\sim}$  for some  $z \leq y$ . By Lemma 22  $x = 1 \cdot x = 1 \cdot z = z$  and hence  $x \leq y$ .

(2) Assume that  $[x, s] \leq [y, 1]$ . Then  $[x, s]_{\sim} \leq [y, 1]_{\sim}$ . By Lemma 21  $(x, s) \sim (z, 1)$  for some  $z \leq y$  implying that  $(x, s) \approx (z, 1)$  whence  $[x, s] \in \overline{Y}$ .

(3) follows from the previous two items.  $\square$

**Lemma 25.** *Let  $[x, s] = [y, p]$ . Then  $[x, ts] = [y, tp]$  for any  $t \in T$ .*

*Proof.* Since  $[x, s]_{\sim} \leq [y, p]_{\sim}$ , we have  $[x, s]_{\sim} = [z, p]_{\sim}$  for some  $z \leq y$  by Lemma 21. We show that  $[x, ts]_{\sim} = [z, tp]_{\sim}$ . Assume that  $(x, s) \rightarrow (z, p)$ . This can be rewritten as  $(x, pq) \rightarrow (q \cdot x, p)$ . But then  $(x, tpq) \rightarrow (q \cdot x, tp)$  which means that  $(x, ts) \rightarrow (z, tp)$ . That  $(x, s) \rightarrow (z, p)$  implies  $(x, ts) \rightarrow (z, tp)$  follows by symmetry. The claim that  $[x, ts]_{\sim} = [z, tp]_{\sim}$  now easily follows by induction. Therefore  $[x, ts]_{\sim} \leq [y, tp]_{\sim}$ . The opposite inequality is shown similarly so that  $[x, ts] = [y, tp]$ , as required.  $\square$

Let  $t \in T$  and  $[y, s] \in X$ . We set  $t * [y, s] = [y, ts]$ . By the preceding lemma this is well defined and thus clearly is an order-preserving left action of  $T$  on  $X$ .

**Lemma 26.**

- (1) *The induced left partial action of  $T$  on  $\overline{Y}$  is isomorphic to the left partial action  $\cdot$  of  $T$  on  $Y$ .*
- (2) *If  $[x, s] \leq t * [y, 1]$  then  $[x, s] = t * [z, 1]$  for some  $z \leq y$ .*

*Proof.* (1) Let  $t \in T$  and  $y \in Y$ . By Lemmas 22 and 23  $[y, t] = [z, 1]$  for some  $z$  if and only if  $t \cdot y$  is defined. Assume that  $t \cdot y$  is defined. Then  $t * [y, 1] = [y, t] = [t \cdot y, 1]$ . Assume that  $t \cdot y$  is undefined. Then  $t * [y, 1] \notin \overline{Y}$  and so the induced partial action is undefined on  $[y, 1]$ .

(2) follows from Lemma 21 and the definition of  $*$ .  $\square$

**Proposition 27.** *Let  $S$  be a left extra proper restriction semigroup. Let  $X$  be the poset and  $*$  be the left action of  $S/\sigma$  on  $X$  obtained by applying the globalization construction to the strong left partial action  $\cdot$  underlying  $S$ . Then for every  $x, y \in \overline{P(S)}$  and  $s \in S/\sigma$  the*

meet  $x \wedge s * y$  exists in  $X$  and the multiplication in the semigroup  $M(S/\sigma, \overline{P(S)})$  can be expressed by the formula:

$$(x, s)(y, t) = (x \wedge s * y, st).$$

*Proof.* By Lemma 26(2) condition (3.2) is satisfied so the statement follows by Lemma 10.  $\square$

Proposition 27 provides a globalized version of Theorem 4 and may be considered an analogue of the McAlister  $P$ -theorem [15, 14] for left extra proper restriction semigroups. A similar statement can be formulated and proved for right extra proper restriction semigroups.

## 5. EMBEDDING OF AN ULTRA $F$ -RESTRICTION MONOID $M(A^*, Y)$ INTO A $W$ -PRODUCT

It is natural to look for conditions on the semigroup  $M(T, Y)$  under which the poset  $X$  constructed in the previous section would be a semilattice and/or the action  $*$  would be ‘nice’. (Recall that in the case when  $M(T, Y)$  is an inverse semigroup the action  $*$  is always by order automorphisms, and  $X$  is a semilattice if and only if  $M(T, Y)$  is  $F$ -inverse).

In this section we show that this can be achieved if  $T$  is a free monoid and  $M(T, Y)$  is ultra  $F$ -restriction. To be precise, our result is formulated as follows:

**Theorem 28.** *Let  $T = A^*$  be the free  $A$ -generated monoid and assume that a left partially defined action  $\cdot$  of  $T$  on a semilattice  $Y$  is given such that the semigroup  $M(T, Y)$  can be formed and is an ultra  $F$ -restriction monoid. Let  $X$  and  $*$  be the poset and the left action of  $T$  on  $X$  constructed in Section 4. Then*

- (1)  $X$  is semilattice.
- (2) The  $W$ -product  $W(T, X)$  may be formed.
- (3)  $M(T, Y)$  embeds into  $W(T, X)$ .

Before we proceed, we point out that Theorem 28 may be applied to the free restriction monoid  $M(Z^*, \mathcal{Y}')$  (see Examples 13 and 14) and also to the monoid  $M(T, E)$  from Section 3.10.

The remainder of this section will be devoted to the proof of Theorem 28. For  $v \in A^*$  by  $|v|$  we denote the length of  $v$ . The empty word is denoted by 1 and we put  $|1| = 0$ . The following two lemmas hold under a milder assumption that  $M(T, Y)$  is an ultra proper restriction semigroup.

### Lemma 29.

- (1) Every  $\sim$ -class  $B$  of  $Y \times T$  has the only representative, which we call canonical, of the form  $(x, w)$  where

$$|w| = \min\{|u| : (y, u) \in B\}.$$

*If  $(x, v)$  is a canonical representative and  $(y, u) \sim (x, v)$  then  $(y, u) = (x \circ t, vt)$  for some  $t \in A^*$ .*

- (2) The equivalences  $\sim$  and  $\approx$  on  $Y \times T$  coincide.

*Proof.* Let  $(y, u) \in B$  and  $v$  be the longest suffix of  $u$  such that  $v \cdot y$  is defined and put  $x = v \cdot y$ . Due to (PDA)  $v' \cdot y$  is defined for any suffix  $v'$  of  $v$ . Let  $w \in A^*$  be such that  $u = vw$  (any of the words  $w$  or  $v$  may be empty). We have  $(y, u) \sim (x, w)$ . Assume that  $(z, v) \sim (x, w)$  and show that there is  $t \in A^*$  such that  $v = x \circ t$  and  $v = wt$ . By the definition of  $\sim$ , there is a sequence

$$(x, w) = (x_0, w_0), (x_1, w_1), \dots, (x_n, w_n) = (z, v)$$

such that for each admissible  $i$  we have  $(x_i, w_i) \rightarrow (x_{i+1}, w_{i+1})$  or  $(x_{i+1}, w_{i+1}) \rightarrow (x_i, w_i)$ . Note that we necessarily have  $(x_1, w_1) \rightarrow (x_0, w_0)$  and so  $(x_1, w_1) = (x_0 \circ t, w_0 t) = (x \circ t, wt)$  for some  $t \in A^*$ . We proceed by induction on  $n$ . Assume that  $(x_i, w_i) = (x \circ t, wt)$ . If  $(x_{i+1}, w_{i+1}) \rightarrow (x_i, w_i)$  it is immediate that  $(x_{i+1}, w_{i+1})$  equals  $(x \circ st, wts)$  for some  $s \in A^*$ . Consider now the case when  $(x_i, w_i) \rightarrow (x_{i+1}, w_{i+1})$ . By assumption we have  $(x_i, w_i) = (x \circ t, wt)$  and then  $(x_{i+1}, w_{i+1}) = (p \cdot x_{i+1}, q)$  where  $p$  is a suffix of  $wt$  and  $q$  is determined by  $qp = wt$ . Note that  $p$  must be a suffix of  $t$  for if we assume that  $p = w't$  with  $w'$  non empty then we obtain that  $w' \cdot x$  is defined (since  $w'p \cdot (t \circ x)$  is defined,  $p \cdot (t \circ x)$  is defined and  $\cdot$  is strong) which contradicts the choice of  $(x, w)$ . This proves that  $(x, w)$  is canonical, that it is unique and the claim about the form of any element in its  $\sim$ -class.

For the second item, we prove that the preorder  $\leq$  on  $(Y \times T)/\sim$  defined before Lemma 21 is in fact an order. Indeed, assume that  $[y, v]_\sim \leq [x, w]_\sim$  and  $[x, w]_\sim \leq [y, v]_\sim$  and assume that  $(x, w)$  is a canonical element. Since  $[y, v]_\sim \leq [x, w]_\sim$  we have that  $(y, v) \sim (z, w)$  for some  $z \leq x$  by Lemma 21. Similarly,  $[x, w]_\sim \leq [z, w]_\sim$  implies that  $(x, w) \sim (x', w)$  for some  $x' \leq z \leq x$ . But the latter is possible only if  $x' = x$  since, as we have proved, any element in  $[x, w]_\sim$  has the form  $(x \circ t, wt)$  for some  $t \in A^*$ .  $\square$

**Example 30.** Let  $F\mathcal{RM}(Z) = M(Z^*, \mathcal{Y}')$  be the free restriction monoid over a set  $Z$ . Then the relation  $\sim$  on  $\mathcal{Y}' \times Z^*$  is given by  $(B, v) \sim (B', v')$  if and only if  $v * B = v' * B'$  where we remind that  $v * B = \{\text{red}(vb) : b \in B\}$ . It follows that  $[B, v]_\sim \mapsto v * B$  defines a bijection between the sets  $(\mathcal{Y}' \times Z^*)/\sim$  and  $\mathcal{Q}'$ . It is easy to see that the order on  $(\mathcal{Y}' \times Z^*)/\sim$  corresponds to the anti-inclusion order on  $\mathcal{Q}'$  under this bijection and the canonical elements of  $\mathcal{Y}' \times Z^*$  are precisely the elements  $(B, v)$  such that either  $v = 1$ , or, otherwise,  $z^{-1} \notin B$  where  $z$  is the last letter of  $v$ . For an arbitrary element  $(B, v)$  the canonical element in its  $\sim$ -class is the element  $(w \cdot B, u)$  where  $w$  is the longest suffix of  $v$  such that  $w^{-1} \in B$  and  $v = uw$ . Recall that the elements of  $\mathcal{X}'$  can be interpreted as finite connected subgraphs in the Cayley graph of  $F\mathcal{G}(Z)$  and the elements of  $\mathcal{Y}'$  as such subgraphs containing the origin (see [19, 20] for details). Then the elements of  $\mathcal{Q}'$  correspond to finite connected subgraphs of the form  $t * A$  where  $A \in \mathcal{Y}'$  and  $t \in Z^*$ . If  $\Gamma$  is such a subgraph then the canonical element  $(B, v)$  corresponding to it under the bijection between  $\mathcal{Q}'$  and  $(\mathcal{Y}' \times Z^*)/\sim$  is determined as follows:  $v$  the vertex of  $\Gamma$  which is closest to the origin (such a vertex exists since the graph is a tree) and  $B = v^{-1} * \Gamma$ .

Recall that the action  $*$  of  $T$  on  $X$  which globalizes  $\cdot$  is given by  $t * [x, w] = [x, tw]$ . Let the map  $\alpha_t : X \rightarrow X$  be given by  $x \mapsto t * x$ .

**Lemma 31.**

- (1) For every  $t \in T$   $\text{ran}(\alpha_t)$  is an order ideal of  $X$ .  
 (2)  $\alpha_t : X \rightarrow \text{ran}(\alpha_t)$  is an order-isomorphism.

*Proof.* Assume that  $[y, v] \leq t * [x, w] = [x, tw]$ . Then  $[y, v] = [z, tw]$  where  $z \leq x$  and so  $[y, v] = t * [z, w]$  which proves the first claim.

It is immediate that  $*$  is order-preserving. So to prove the second claim, we assume that  $t * [x, w] \leq t * [y, v]$  and show that  $[x, w] \leq [y, v]$ . Let  $q$  be the longest suffix of  $tw$  such that  $q \cdot x$  is defined and assume that  $tw = pq$ . Then

$$[x, tw] = [x, pq] = [q \cdot x, p]$$

and the element  $(q \cdot x, p)$  is canonical. Similarly,  $[y, tv] = [q' \cdot y, p']$  with the element  $(q' \cdot y, p')$  being canonical where  $q'$  is the longest suffix of  $tu$  such that  $q' \cdot y$  is defined. Assume that  $tv = p'q'$ . Since  $[q \cdot x, p] \leq [q' \cdot y, p']$  there is some  $z \leq y$  such that

$$[q \cdot x, p] = [q' \cdot z, p'].$$

Canonicity of  $(q \cdot x, p)$  implies that

$$(q' \cdot z, p') = ((q \cdot x) \circ m, pm)$$

for some  $m \in T$ . It follows that

$$q \cdot x = mq' \cdot z \text{ and } pm = p'.$$

Then we have  $tw = pq$  and  $tv = pmq'$ . Consider two possible cases:

*Case 1.* Assume that  $|p| \geq |t|$ . Then  $|q| \leq |w|$  and  $|mq'| \leq |v|$  which implies that  $w = rq$  and  $v = rmq'$  for some suffix  $r$  of  $p$ . Then we have:

$$[x, w] = [x, rq] = [q \cdot x, r] = [mq' \cdot z, r] = [q' \cdot z, rm] \leq [q' \cdot y, rm] = [y, rmq'] = [y, v].$$

*Case 2.* Assume that  $|p| < |t|$ . Then  $|q| > |w|$  and  $|mq'| > |v|$ . Thus  $q = kw$  and  $mq' = kv$  for some suffix  $k$  of  $t$ . Since  $q \cdot x$  defined then  $w \cdot x$  is defined and we have:

$$w \cdot x = (q \cdot x) \circ k = (mq' \cdot z) \circ k \leq (mq' \cdot y) \circ k = (kv \cdot y) \circ k = v \cdot y.$$

Therefore  $[x, w] = [w \cdot x, 1] \leq [v \cdot y, 1] = [y, v]$ , which completes the proof.  $\square$

From now on we assume that  $M(T, Y)$  is an ultra  $F$ -restriction monoid. For  $v \in T$  let  $d_v$  and  $r_v$  denote the top elements of  $\text{dom}(\varphi_t)$  and  $\text{ran}(\varphi_t)$ , respectively.

**Lemma 32.**  *$X$  is a semilattice. The meet on  $X$  is calculated by the rule:*

$$(5.1) \quad [e, v] \wedge [f, u] = [v' \cdot (e \wedge d_{v'}) \wedge u' \cdot (f \wedge d_{u'}), k]$$

where  $k$  is the longest common prefix of  $v$  and  $u$  and  $v = kv'$ ,  $u = ku'$ .

*Proof.* Let  $[e, v], [f, u] \in X$ . Let  $k$  be the longest common prefix of  $v$  and  $u$  and assume that  $v = kv'$  and  $u = ku'$ . We put

$$g = v' \cdot (e \wedge d_{v'}) \wedge u' \cdot (f \wedge d_{u'})$$

and show that  $[g, k] = [e, v] \wedge [f, u]$ . To verify that  $[e, v] \geq [g, k]$  it is enough to show that  $[g, k] = [e', v]$  where  $e' \leq e$ . Since  $g \leq v' \cdot (e \wedge d_{v'})$  then  $g \circ v'$  is defined and  $g \circ v' \leq e \wedge d_{v'}$ . Therefore

$$[g, k] = [g \circ v', kv'] \leq [e \wedge d_{v'}, v] \leq [e, v].$$

Similarly,  $[g, k] \leq [f, u]$ .

Suppose that  $[h, t] \leq [e, v], [f, u]$  where we assume that the element  $(h, t)$  is canonical. Then there are  $e' \leq e$  and  $f' \leq f$  such that  $[h, t] = [e', v] = [f', u]$ . Since  $(h, t)$  is canonical, by Lemma 29(1) we have

$$e' = h \circ p, v = tp, f' = h \circ q, u = tq$$

for some  $p, q \in T$ . Hence  $t$  is a common prefix of  $v$  and  $u$  and so  $k = tl$  for some  $l \in T$  by maximality of  $k$ . Then also  $p = lv'$  and  $q = lu'$ . Since  $h \circ p$  is defined then  $h \circ l$  and  $(h \circ l) \circ v'$  are defined and  $e' = h \circ p = (h \circ l) \circ v'$  by (PDA). Similarly,  $h \circ q = (h \circ l) \circ u'$ . Then

$$[e', v] = [h \circ lv', tl v'] = [h \circ l, tl] = [h \circ l, k].$$

Since  $h \circ p \leq e$  and  $h \circ p = (h \circ l) \circ v' \leq d_{v'}$ , we obtain  $h \circ p \leq e \wedge d_{v'}$ . Hence

$$h \circ l = v' \cdot (h \circ p) \leq v' \cdot (e \wedge d_{v'}).$$

Similarly one shows that  $h \circ l \leq u' \cdot (f \wedge d_{u'})$ . It follows that  $h \circ l \leq g$  and so  $[h, t] = [h \circ l, k] \leq [g, k]$ , which completes the proof.  $\square$

It follows from Lemmas 31 and 32 that we may form the  $W$ -product  $W(T, X)$ . As is shown in Subsection 3.2 we have  $W(T, X) = M(T, X)$ . To complete the proof of Theorem 28 we argue that  $M(T, \bar{Y})$  embeds into  $W(T, X)$ . Let  $\hat{\cdot}$  be the left partially defined action of  $T$  on  $\bar{Y}$  isomorphic to  $\cdot$  via  $[y, 1] \mapsto y$ . Let further  $\varphi : T \rightarrow \mathcal{I}(\bar{Y})$  be the homomorphism defining  $\hat{\cdot}$  and  $\alpha : T \rightarrow \mathcal{I}(X)$  the homomorphism defining  $*$ . Then for every  $t \in T$   $\varphi_t$  is a restriction of  $\alpha_t$ , which implies that  $\varphi_t^{-1}$  is a restriction of  $\alpha_t^{-1}$ . This means that the right partially defined action  $\bullet$  reverse to  $*$  is an extension of the right partially defined action  $\circ$  reverse to  $\hat{\cdot}$ . In particular for every  $t \in T$  and  $[y, 1] \in \bar{Y}$  if  $[y, 1] \circ t$  is defined then  $[y, 1] \bullet t$  is defined and  $[y, 1] \circ t = [y, 1] \bullet t$ . Therefore,

$$(5.2) \quad M(T, \bar{Y}) = \{([y, 1], t) \in \bar{Y} \times T : [y, 1] \circ t \text{ is defined}\} = \\ \{([y, 1], t) \in \bar{Y} \times T : [y, 1] \bullet t \text{ is defined and } [y, 1] \bullet t \in \bar{Y}\} \subseteq M(T, X) = W(T, X).$$

Applying Proposition 27 it is now immediate that  $M(T, \bar{Y})$  is a  $(2, 1, 1)$ -subalgebra of  $M(T, X)$ .

**Example 33.** Let  $F\mathcal{R}M(Z) = M(Z^*, \mathcal{Y}')$  be the free restriction monoid. As already remarked, the semilattice  $(\mathcal{Y}' \times Z^*) / \sim$  is order isomorphic to  $\mathcal{Q}'$ . It is easy to see that the action of  $Z^*$  on  $(\mathcal{Y}' \times Z^*) / \sim$  which globalizes  $\cdot$  is isomorphic to the action  $*$  of  $Z^*$  on  $\mathcal{Q}'$  given by  $v * E = \{\text{red}(ve) : e \in E\}$ . It follows that the monoid  $W(T, X)$  from Theorem 28(2) is isomorphic to the monoid  $W(Z^*, \mathcal{Q}')$  and the embedding of  $F\mathcal{R}M(Z)$  into  $W(A^*, \mathcal{Q}')$  produced by the proof of Theorem 28 coincides with the (left hand version of the) embedding constructed by Szendrei in [19]. In particular, the monoid  $M(T, \bar{Y})$  from

(5.2) is in this setting precisely the Szendrei's model  $F_W\mathcal{RM}(Z)$  of the free restriction monoid as is defined in Section 2.5

**Remark 34.** We remark that Lemma 32 and thus also Theorem 28 can be formulated and proved in a slightly more general setting. Namely, it is enough to assume that  $M(T, X)$  is an ultra proper restriction semigroup, and  $\text{dom}(\varphi_t)$  is a principal order ideal for all  $t \neq 1$ . Examples where this setting arises are the free restriction semigroup  $F\mathcal{RS}(Z)$  and also the semigroup  $M(T, E)$  from Section 3.11. The resulting construction of embedding of  $M(T, E)$  into a  $W$ -product generalizes Szendrei's embedding of  $F\mathcal{RS}(Z)$  into a  $W$ -product [19].

Applying Theorem 28 to the settings of Section 3.10 we deduce the following result strengthening the main result of [19].

**Theorem 35.**

- (1) Every restriction monoid  $S = \langle A \rangle$  has an ultra  $F$ -restriction (ample) cover  $M(A^*, P(S))$  which  $(2, 1, 1)$ -embeds into a  $W$ -product  $W(A^*, X)$ .
- (2) Every restriction semigroup  $S = \langle A \rangle$  has an ultra proper (ample) cover  $M(A^*, P(S))$  which  $(2, 1, 1)$ -embeds into a  $W$ -product  $W(A^*, X')$ .

*Proof.* The first claim is immediate by Lemma 15 and Theorem 28. For the second claim we let  $S = \langle A \rangle$  to be a restriction semigroup and recall that the semigroup  $M(A^*, P(S))$  constructed in Section 3.11 is an ultra proper cover of  $S$  and is a  $(2, 1, 1)$ -subalgebra of the ultra  $F$ -restriction monoid  $M(A^*, P(S)^1)$ . The latter can be  $(2, 1, 1)$ -embedded into a  $W$ -product  $W(A^*, X')$  by Theorem 28.  $\square$

## 6. AN EMBEDDING OF A RESTRICTION SEMIGROUP INTO A QUOTIENT OF $W(A^*, X)$

Let  $S = \langle A \rangle$  be a restriction monoid with respect to  $E$ . Let, further,  $M(A^*, E)$  be the ultra  $F$ -restriction cover of  $S$  from Section 3.10 and  $W(A^*, X)$  the  $W$ -product produced by the proof of Theorem 28. In this final section we construct a projection separating  $(2, 1, 1)$ -congruence  $\kappa$  on  $W(A^*, X)$  such that  $S$   $(2, 1, 1)$ -embeds into  $W(A^*, X)/\kappa$ . This yields a new and simpler proof of the main result of [20]. We set  $W = W(A^*, X)$ .

**Lemma 36.** *Let  $([e, c], d) \in W$  where the element  $(e, c)$  is canonical. Then either  $c$  is a prefix of  $d$  or  $d$  is a prefix of  $c$ .*

*Proof.* By the definition of  $W$  there is some  $[f, b] \in X$  such that  $[e, c] = d * [f, b]$ . Then  $[e, c] = [f, db]$  and so  $(f, db) = (e \circ g, cg)$  for some  $g \in T$  by canonicity of  $(e, c)$  applying Lemma 29(1). Hence  $db = cg$ , which implies the needed statement.  $\square$

We define the sets  $W_1$  and  $W_2$  as follows:

$$\begin{aligned} W_1 &= \{([e, p], pq) \in W : (e, p) \text{ canonical}, q \in T\}; \\ W_2 &= \{([e, pq], p) \in W : (e, pq) \text{ canonical}, q \in T \setminus \{1\}\}. \end{aligned}$$

By Lemma 36  $W = W_1 \cup W_2$  and it is clear that  $W_1 \cap W_2 = \emptyset$ . Define an auxiliary relation  $\gamma$  on  $W_1$  by setting  $x \gamma y$  if and only if

$$x = ([e, p], pq), y = ([e, p], pr) \text{ and } e\bar{q} = e\bar{r},$$

where the element  $(e, p)$  is canonical. For the element  $x = ([e, p], pq) \in W_1$  where  $(e, p)$  is canonical we put

$$\text{inv}(x) = e\bar{q}.$$

We now define a relation  $\kappa$  on  $W$  by putting for  $x, y \in W$ :  $x \kappa y$  if and only if  $x = y$  or  $x, y \in W_1$  and  $x \gamma y$ .

**Lemma 37.** *The relation  $\kappa$  is a  $(2, 1, 1)$ -congruence.*

*Proof.* It is immediate that it is an equivalence relation. Let  $x \kappa y$ . We may assume that  $x, y \in W_1$  and that  $x = ([e, p], pq)$ ,  $y = ([e, p], pr)$  where  $(e, p)$  is a canonical element and  $e\bar{q} = e\bar{r}$ . It is immediate that  $\rho(x) = ([e, p], 1) = \rho(y)$ . We show that  $\lambda(x) = \lambda(y)$ . We have

$$\lambda(x) = ([e \circ q, 1], 1), \quad \lambda(y) = ([e \circ r, 1], 1).$$

The elements  $(e \circ q, 1)$  and  $(e \circ r, 1)$  are obviously canonical. The needed equality follows from

$$e \circ q = \lambda(e\bar{q}) = \lambda(e\bar{r}) = e \circ r.$$

We now show that for every  $z \in W$  we have  $xz \kappa xy$  and  $zx \kappa zy$ . Let  $z = ([f, a], b)$ .

*Case 1.* Show that  $xz \kappa yz$ . Applying (5.1) we have

$$(6.1) \quad xz = ([e, p] \wedge [f, pqa], pqb) = ([e\rho(\bar{q}af), p], pqb).$$

Note that

$$\begin{aligned} e\rho(\bar{q}af) &= \rho(e\rho(\bar{q}af)) && \text{(by (RS2))} \\ &= \rho(e\bar{q}af) && \text{(by (RS4))} \\ &= \rho(e\bar{q}\rho(\bar{a}f)) && \text{(by (RS4))} \end{aligned}$$

and similarly  $e\rho(\bar{r}af) = \rho(e\bar{r}\rho(\bar{a}f))$ . It follows that  $e\rho(\bar{q}af) = e\rho(\bar{r}af)$ , let us denote this element by  $e'$ . Then

$$xz = ([e', p], pqb) \in W_1 \text{ and } yz = ([e', p], prb) \in W_1.$$

Let  $p = p_1 p_2$  be such a factorization that the element  $(p_2 \cdot e', p_1)$  is canonical. We obtain

$$\begin{aligned} \text{inv}(xz) &= (p_2 \cdot e') \overline{p_2 q b} \\ &= \rho(\overline{p_2 e'}) \overline{p_2 q b} \\ &= \overline{p_2 e' q} \bar{b} && \text{(by (RS7))} \end{aligned}$$

and similarly  $\text{inv}(yz) = \overline{p_2 e' r} \bar{b}$ . Our assumption that  $x \kappa y$  and  $e' \leq e$  imply that  $e'\bar{q} = e'\bar{r}$ . Therefore,  $\text{inv}(xz) = \text{inv}(yz)$  and so  $xz \kappa yz$ .

*Case 2.* Show that  $zx \kappa zy$ .

$$zx = ([f, a], b)([e, p], pq) = ([f, a] \wedge [e, bp], bpq).$$

Let  $c$  be the longest common prefix of  $a$  and  $bp$  so that  $a = ca'$  and  $bp = cb'$ . Then

$$[f, a] \wedge [e, bp] = [\rho(\overline{a'}f)\rho(\overline{b'}e), c]$$

and thus

$$zx = ([\rho(\overline{a'}f)\rho(\overline{b'}e), c], cb'q) \in W_1.$$

Substituting  $q$  with  $r$  in the above calculation for  $zx$  it follows that

$$zy = ([\rho(\overline{a'}f)\rho(\overline{b'}e), c], cb'r) \in W_1.$$

Let  $c = kl$  where the element  $(l \cdot (\rho(\overline{a'}f)\rho(\overline{b'}e)), k)$  is canonical. Then

$$\begin{aligned} \text{inv}(zx) &= \rho(\overline{l}\rho(\overline{a'}f)\rho(\overline{b'}e))\overline{l\overline{b'}q} \\ &= \overline{l}\rho(\overline{a'}f)\rho(\overline{b'}e)\overline{b'q} && \text{(by (RS7) since } \overline{l}\rho(\overline{a'}f)\rho(\overline{b'}e) \leq \overline{l}) \\ &= \overline{l}\rho(\overline{a'}f)\overline{b'e}\overline{q} && \text{(by (RS7) since } \overline{b'e} \leq \overline{b'}) \end{aligned}$$

Hence  $\text{inv}(zx) = \text{inv}(zy)$  so that  $zx \kappa zy$ , as required. We have verified that  $\kappa$  is a  $(2, 1, 1)$ -congruence on  $W$ . That  $\kappa$  is projection separating is immediate from its definition.  $\square$

**Theorem 38** (Szendrei [20]). *Every restriction semigroup is  $(2, 1, 1)$ -embeddable into a  $(2, 1, 1)$ -morphic image of a  $W$ -product of a semilattice by a monoid.*

*Proof.* Let  $S = \langle A \rangle$  be a restriction semigroup and  $M(A^*, E)$  the ultra proper cover of  $S$  from Section 3.11. As is explained in Section 3.11, by adjoining an identity to  $M(A^*, E)$  we obtain an ultra  $F$ -restriction monoid  $M(A^*, E^1)$ , and  $M(A^*, E)$  is a  $(2, 1, 1)$ -subalgebra of  $M(A^*, E^1)$ . Let  $W = W(A^*, X)$  be the  $W$ -product produced by the proof of Theorem 28 out of the monoid  $M(A^*, E^1)$  and  $\kappa$  be the congruence on  $W$  constructed in this section. It is immediate from the definition of  $\kappa$  that for  $([x, 1], s), ([y, 1], t) \in M(A^*, \overline{E})$  we have  $([x, 1], s) \kappa ([y, 1], t)$  if and only if  $x\overline{s} = y\overline{t}$ . Since the quotient of  $M(A^*, \overline{E})$  over this congruence is isomorphic to  $S$ , this yields that  $S$  embeds into  $W/\kappa$ .  $\square$

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